

Solutions 4

Exercise 5.16

$$p(\mathbf{x}; \theta) = \frac{1}{(2\pi)^{N/2} \det^{1/2}(\mathbf{C})} \exp\left\{-\frac{1}{2}(\mathbf{x} - \mu)^T \mathbf{C}^{-1}(\mathbf{x} - \mu)\right\}$$

where

$$\det(\mathbf{C}) = N - 1 + \sigma^2 - (-\mathbf{1})^T(-\mathbf{1}) = \sigma^2$$

and

$$\mathbf{C}^{-1} = \begin{bmatrix} \frac{1}{\sigma^2} & \frac{1}{\sigma^2} \mathbf{1}^T \\ \frac{1}{\sigma^2} \mathbf{1} & \left(\mathbf{I} - \frac{\mathbf{1}\mathbf{1}^T}{N-1+\sigma^2}\right)^{-1} \end{bmatrix}$$

We can get $\left(\mathbf{I} - \frac{\mathbf{1}\mathbf{1}^T}{N-1+\sigma^2}\right)^{-1} = \mathbf{I} + \frac{\mathbf{1}\mathbf{1}^T}{\sigma^2}$

$$\mathbf{C}^{-1} = \frac{1}{\sigma^2} \begin{bmatrix} 1 & \mathbf{1}^T \\ \mathbf{1} & \sigma^2 \mathbf{I} + \mathbf{1}\mathbf{1}^T \end{bmatrix}$$

$$\begin{aligned} (\mathbf{x} - \mu)^T \mathbf{C}^{-1}(\mathbf{x} - \mu) &= \frac{1}{\sigma^2} [x[0] - N\mu, x[1], \dots, x[N-1]] \begin{bmatrix} 1 & \mathbf{1}^T \\ \mathbf{1} & \sigma^2 \mathbf{I} + \mathbf{1}\mathbf{1}^T \end{bmatrix} \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix} \\ &= \frac{1}{\sigma^2} \left[\left(\sum_{n=0}^{N-1} x[n] - N\mu \right)^2 + \sigma^2 \sum_{n=0}^{N-1} x[n]^2 \right] \\ &= \frac{N^2}{\sigma^2} (\bar{x} - \mu)^2 + \sum_{n=0}^{N-1} x^2[n] \end{aligned}$$

Thus,

$$p(\mathbf{x}; \boldsymbol{\theta}) = \frac{1}{(2\pi)^{N/2} \sigma} \exp\left\{-\frac{1}{2} \left[\frac{N^2}{\sigma^2} (\bar{x} - \mu)^2 + \sum_{n=0}^{N-1} x^2[n] \right]\right\}$$

\bar{x} is the sufficient statistic for $\boldsymbol{\theta}$.

Exercise 5.18

$$p(\mathbf{x}, \boldsymbol{\theta}) = \frac{1}{(\theta_2 - \theta_1)^N} u(\min x[n] - \theta_1) u(\max x[n] - \theta_2)$$

The sufficient statistic is

$$\mathbf{T}(\mathbf{x}) = \begin{bmatrix} \min x[n] \\ \max x[n] \end{bmatrix}$$

Exercise 6.1

Since $\mathbb{E}\{x[n]\} = Ar^n$, we have $\mathbf{H} = (1, r, \dots, r^{N-1})^T$ and $\mathbf{C}^{-1} = \frac{1}{\sigma^2}\mathbf{I}$. Thus,

$$\hat{A} = \frac{\mathbf{H}^T \mathbf{C}^{-1} \mathbf{x}}{\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H}} = \frac{\sum_{n=0}^{N-1} x[n] r^n}{\sum_{n=0}^{N-1} r^{2n}}$$

$$\text{var}(\hat{A}) = (\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H})^{-1} = \frac{\sigma^2}{\sum_{n=0}^{N-1} r^{2n}}$$

$\text{var}(\hat{A}) \rightarrow 0$ only if $|r| \geq 1$.

Exercise 6.4

(a) For Laplacian distribution, we have $\mathbb{E}\{x[n]\} = \mu$ and $\text{var}(x[n]) = 2$. Thus, we have $\mathbf{H} = \mathbf{1}$, $\mathbf{C}^{-1} = \frac{1}{2}\mathbf{I}$ and

$$\hat{\mu} = \frac{\mathbf{H}^T \mathbf{C}^{-1} \mathbf{x}}{\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H}} = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$$

(b) For Gaussian distribution, the same with (a), we have $\hat{\mu} = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$.

In the Gaussian case, the BLUE is also MVU estimator, but not true in the Laplacian case.

Exercise 6.5

Let $y = \ln x$, we can get

$$\begin{aligned} \mathbb{E}(x) &= \int_0^\infty \frac{x}{\sqrt{2\pi x}} \exp\left[-\frac{1}{2}(\ln x - \theta)^2\right] dx \\ &= \int_{-\infty}^\infty \exp\left[-\frac{1}{2}(y - \theta)^2 + y\right] dy \\ &= \int_{-\infty}^\infty \exp\left[-\frac{1}{2}(y - (\theta + 1))^2\right] \exp(\theta + \frac{1}{2}) dy \\ &= \exp(\theta + \frac{1}{2}) \end{aligned}$$

Also, $y \sim \mathcal{N}(0, 1)$. The BLUE is $\hat{\theta} = \frac{1}{N} \sum_{n=0}^{N-1} y[n] = \frac{1}{N} \sum_{n=0}^{N-1} \ln x[n]$.

Exercise 6.9

$\mathbf{H} = (1, \cos 2\pi f_1, \dots, \cos 2\pi(N-1)f_1)^T$ and $\mathbf{C}^{-1} = \frac{1}{\sigma^2}\mathbf{I}$

$$\hat{A} = \frac{\mathbf{H}^T \mathbf{C}^{-1} \mathbf{x}}{\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H}} = \frac{\sum_{n=0}^{N-1} x[n] \cos 2\pi f_1 n}{\sum_{n=0}^{N-1} \cos^2 2\pi f_1 n}$$

$$\text{var}(\hat{A}) = (\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H})^{-1} = \frac{\sigma^2}{\sum_{n=0}^{N-1} \cos^2 2\pi f_1 n}$$

Obviously, when $\cos^2 2\pi f_1 n = 1$, we can obtain minimum $\text{var}(\hat{A})$. Thus, $f_1 = 0$.

Exercise 6.14

$$\begin{aligned}\mathbb{E}[w[n]] &= (1 - \epsilon)\mathbb{E}[w_B[n]] + \epsilon\mathbb{E}[w_I[n]] = 0 \\ \mathbb{E}[w^2[n]] &= \int_{-\infty}^{+\infty} w^2 p(w) dw \\ &= (1 - \epsilon) \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma_B^2}} e^{-\frac{w^2}{2\sigma_B^2}} dw + \epsilon \int_{-\infty}^{+\infty} w^2 \frac{1}{\sqrt{w\pi\sigma_I^2}} e^{-\frac{w^2}{2\sigma_I^2}} \\ &= (1 - \epsilon)\sigma_B^2 + \epsilon\sigma_I^2 \\ \text{var}(w[n]) &= (1 - \epsilon)\sigma_B^2 + \epsilon\sigma_I^2\end{aligned}$$

Then, we let $y[n] = w^2[n] + z[n]$, where $z[n]$ is the noise with zero-mean, and its variance matrix is $\sigma_z^2 \mathbf{I}$. Here, $\mathbf{H} = \mathbf{1}$ and $\mathbf{C} = \sigma_z^2 \mathbf{I}$.

We can find the BLUE of σ^2 first as follows:

$$\hat{\sigma}^2 = \frac{\mathbf{H}^T \mathbf{C}^{-1} \mathbf{x}}{\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H}} = \frac{1}{N} \sum_{n=0}^{N-1} w^2[n]$$

Then, using the result in Prob. 6.12, we can get

$$\begin{aligned}\hat{\sigma}_I^2 &= \frac{\hat{\sigma}^2 - (1 - \epsilon)\sigma_B^2}{\epsilon} \\ &= \frac{\frac{1}{N} \sum_{n=0}^{N-1} w^2[n] - (1 - \epsilon)\sigma_B^2}{\epsilon}\end{aligned}$$

Exercise 7.1

$$\begin{aligned}\frac{\partial \ln p(\mathbf{x}; A)}{\partial A} &= -\frac{N}{2A} + \frac{1}{A} \sum_{n=0}^{N-1} (x[n] - A) + \frac{1}{2A^2} \sum_{n=0}^{N-1} (x[n] - A)^2 \\ \frac{\partial \ln p^2(\mathbf{x}; A)}{\partial A^2} &= \frac{N}{2A^2} - \frac{1}{A^2} \sum_{n=0}^{N-1} x[n] - \frac{1}{A^2} \sum_{n=0}^{N-1} (x[n] - A) - \frac{1}{A^3} \sum_{n=0}^{N-1} (x[n] - A)^2\end{aligned}$$

Since $\mathbb{E}\{x[n]\} = A$ and $\text{var}(x[n]) = A$, we have

$$\begin{aligned}\mathbb{E}\left[\frac{\partial \ln p^2(\mathbf{x}; A)}{\partial A^2}\right] &= -\frac{N}{2A^2} - \frac{N}{A} \\ \text{var}(\hat{A}) &\geq \frac{A^2}{N(A + 1/2)}\end{aligned}$$

Exercise 7.5

According to Chebychev's inequality, we have $\Pr\{|\hat{\theta} - \theta| > \epsilon\} \leq \frac{\text{var}(\hat{\theta})}{\epsilon^2}$, where $\hat{\theta} = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$ and $\text{var}(\hat{\theta}) = \frac{\sigma^2}{N}$. Thus,

$$\lim_{N \rightarrow \infty} \Pr\{|\hat{\theta} - \theta| > \epsilon\} \leq \lim_{N \rightarrow \infty} \frac{\sigma^2}{\epsilon^2 N} = 0$$

Exercise 7.6

Linearizing g about the true value of θ , i.e. θ_0 , yields

$$\alpha = g(\theta) \approx g(\theta_0) + \left. \frac{dg}{d\theta} \right|_{\theta=\theta_0} (\theta - \theta_0)$$

$$\hat{\alpha} - \alpha = g(\hat{\theta}) - g(\theta_0) \approx [g(\theta_0) + \left. \frac{dg}{d\theta} \right|_{\theta=\theta_0} (\hat{\theta} - \theta_0)] - g(\theta_0) = \left. \frac{dg}{d\theta} \right|_{\theta=\theta_0} (\hat{\theta} - \theta_0)$$

$$\Pr\{|\hat{\alpha} - \alpha| > \epsilon\} = \Pr\left\{\left| \left. \frac{dg}{d\theta} \right|_{\theta=\theta_0} (\hat{\theta} - \theta_0) \right| > \epsilon\right\} = \Pr\left\{|\hat{\theta} - \theta_0| > \frac{\epsilon}{\left| \left. \frac{dg}{d\theta} \right|_{\theta=\theta_0} \right|}\right\}$$

Since $\hat{\theta}$ is consistent for θ , we have $\lim_{N \rightarrow \infty} \Pr\{|\hat{\theta} - \theta| > \epsilon\} = 0$ as long as $dg/d\theta$ is bounded. Hence, $\Pr\{|\hat{\alpha} - \alpha| > \epsilon\} \rightarrow 0$ as $N \rightarrow \infty$

Exercise 7.16

$$\begin{aligned} \text{var}(\hat{A}) &= \text{var}(x[0]) = \text{var}(w[0]) \\ &= \int_{-\infty}^{\infty} u^2 \frac{1}{2} e^{-|u|} du \\ &= \int_0^{\infty} u^2 e^{-u} du \\ &= 2 \end{aligned}$$

$$\frac{dp(u)}{du} = \begin{cases} -\frac{1}{2}e^{-u}, & u > 0 \\ \frac{1}{2}e^u, & u < 0 \end{cases}$$

$$\begin{aligned} \text{var}(\hat{A}) &\geq \int_{-\infty}^{\infty} \left(\frac{dp(u)}{du}\right)^2 / p(u) du \\ &= \int_{-\infty}^{\infty} \frac{1}{2} e^{-|u|} du \\ &= 1 \end{aligned}$$

Thus, MLE doesn't attain the CRLB as $N \rightarrow \infty$.

Exercise 7.19

$$p(\mathbf{x}; f_0) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n] - \cos 2\pi f_0 n)^2\right\}$$

To find MLE, we should minimize

$$\begin{aligned} \sum_n (x[n] - \cos 2\pi f_0 n)^2 &= \sum_n x^2[n] - 2 \sum_n x[n] \cos 2\pi f_0 n + \sum_n \cos^2 2\pi f_0 n \\ &= \sum_n x^2[n] - 2 \sum_n x[n] \cos 2\pi f_0 n + \sum_n \frac{1 - \cos 4\pi f_0 n}{2} \\ &\approx \sum_n x^2[n] - 2 \sum_n x[n] \cos 2\pi f_0 n + \frac{N}{2} \end{aligned}$$

where $\sum_n \cos 4\pi f_0 n \approx 0$.

To maximize $\sum_n x[n] \cos 2\pi f_0 n$ over $0 < f_0 < \frac{1}{2}$, we use Newton-Raphson iteration,

$$g(f_0) = \sum_n x[n] \cos 2\pi f_0 n$$

$$dg/df_0 = - \sum_n (2\pi n) x[n] \sin 2\pi f_0 n$$

$$d^2g/df_0^2 = - \sum_n (2\pi n)^2 x[n] \cos 2\pi f_0 n$$

Thus,

$$f_0^{(k+1)} = f_0^{(k)} - \frac{dg/df_0}{d^2g/df_0^2} \Big|_{f_0=f_0^{(k)}} = f_0^{(k)} - \frac{\sum_n (2\pi n) x[n] \sin 2\pi f_0 n}{\sum_n (2\pi n)^2 x[n] \cos 2\pi f_0 n} \Big|_{f_0=f_0^{(k)}}$$

The code for grid search:

```
clear all;close all;clc;
M = 300;
N = 10; f_0 = 0.25; sigam2 = 0.01;
n = 0:1:(N-1);
n = n';
u = cos(2*pi*f_0*n);
theta = ones(N,1)*sqrt(sigam2);
f_e = 0:0.0005:0.5;
for i=1:M
x = normrnd(u,theta); % generate data
for j = 1:length(f_e)
val_re(1,j) = x'*cos(2*pi*n*f_e(j));
end
[a,b] = max(val_re);
result_f(1,i) = f_e(1,b);
end
plot(1:1:M,result_f);
f_mean= mean(result_f)
f_var = var(result_f)
```

The code for Newton-Raphson:

```
clear all;close all;clc;
M = 300;
N = 10; f_0 = 0.25; sigam2 = 0.01;
n = 0:1:(N-1);
n = n';
u = cos(2*pi*f_0*n);
theta = ones(N,1)*sqrt(sigam2);
f_init = 0.24; % the initial point
can only set in the interval
[0.23,0.27], otherwise it diverges.
for i=1:M
x = normrnd(u,theta); % generate data
f_k = f_init;
for j=1:20 %control the iteration
ud_val = (x'*((2*pi*n).*(sin(2*pi*f_k*n))))...
/(x'*(((2*pi*n).^2).*(cos(2*pi*f_k*n))));
f_k = f_k - ud_val;
end
result_f(1,i) = f_k;
end
plot(1:1:M,result_f);
f_mean = mean(result_f)
f_var = var(result_f)
```

The code for Appendix 7A:

```
clear all;close all;clc;
M = 30000; % M times experiment
N = 100; % N data in each experiment
A = 0.25; % The real A

A_m = zeros(1,M);
for i=1:M
%% Step 1
u1 = rand(N,1);
u2 = rand(N,1);

%% Step 2
w = sqrt(A) * sqrt(-2*log(u1)).* cos(2*pi*u2);

%% Step 3
x = A * ones(N,1) + w;

%% Step 4
A_m(1,i) = -0.5 + sqrt(mean(x.^2) + (1/4));
end
[n,x]= hist(A_m,20);
m_A = mean(A_m); % The mean
var_A = var(A_m); % The variance
plot(x,n/M);
grid on;
title(' The pdf of the estimat A ');
```